GB20602 - Programming Challenges

Week 7 - String Problems

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Topics for this week

Topics for this week:

- String Matching;
  - Naive search;
  - KMP;
  - Z-Algorithm;
- Strings algorithms with DP;
  - Edit Distance
  - Common substring
  - Palindromes
- Suffix Trie
  - Suffix Tree; – idea
  - Suffix Array; – Implementation

Why Study String Problems?

The manipulations of string is a common task in real life applications such as:

- Analysis of Bioinformatics Gene Data;
- Pre-processing/wrangling, of API data (ex: JSON)
- Text processing from human interfaces (natural language)

Characteristics of String Problems

- "Parsing" of inputs with special rules;
- Using Dynamic Programming for finding patterns;
- Special data structures for storing patterns;

Part I: String Matching

The String Matching Problem

Definition

Given a string T (also called **text**), we want to test if the substring P (also called **pattern**) exists in T.

If P exists in T, we want to know the **index** of the start of P in T.

Example:

```
T: STEVEN EVENT

P: EVE              indexes: 2 and 7
P: EVENT            indexes: 7
P: EVENING          indexes: -1 or NULL
```

String Matching and Libraries

String Matching with the standard functions

- C/C++: strstr(T,P) or T.find(P)
- Python: T.find(P)
- Java: T.indexOf(P)

Using the standard library is usually bug-free, but sometimes you need to code string search by hand:

- Specific Matching Function (ex: "1" == "l", "0" == "O");
- Match in multiple directions (graph, grind);
- Match multiple strings at once;
- etc...

String Matching: Complete Search

For every character T_i, test if P begins at that position.

```
for (int i = 0; i < |T|; i++)
  bool match = true;
  for (int j = 0; j < |P| && match; j++)
    if (i+j >= |T| || P[j] != T[i+j])
      match = false;
  if (match)
    printf("Match P at index %d\n", i);
```

Number of Steps:

- Average case: $O(|T|)$ – For natural T and small P;
- Worst case: $O(|T| \times |P|)$;
  - T = AAAAAAAAAAAAB
  - P = AAAAAAB

The Knuth-Morris-Pratt (KMP) Algorithm

- Complete Search can be very expensive if the prefix of P happens many times in T.
- In 1977, Knuth, Morris and Pratt developed an algorithm that **uses these prefixes** to realize fast string matching.

Basic Idea

- The KMP algorithm identifies "borders" in the partial match between P and T.
- These borders are characterized by identical prefixes and sufixes in the T-P match.
- The algorithm uses these matches to advance the indexes of T and P, greatly reducing the number of comparisons.

The KMP algorithm is O(P+T).

KMP Algorithm – Simulation

```
              1         2         3         4         5
    012345678901234567890123456789012345678901234567890
T = I DO NOT LIKE SEVENTY SEV BUT SEVENTY SEVENTY SEVEN
P = SEVENTY SEVEN
// for i from 0 to 13, KMP works like full search

                  SEVENTY SEVEN
// Here, the collision is at i=25, j = 11, But because "SEV" is
// a "border", i stays the same and j is rewinded to 3

                                  SEVENTY SEVEN
// Here we find a match with i=43, j=13; SEVEN is a border, so j
// is rewinded to 5, and i is kept the same. The algorithm
// continues matching at i=44, j=5 ("T")

                                          SEVENTY SEVEN
// KMP finds a second match
```

KMP Algorithm – Rewind Array

To avoid repeated matches, the KMP algorithm builds a **rewind table** *b* (back).

```
       0  1  2  3  4  5  6  7  8  9  0  1  2  3
P =    S  E  V  E  N  T  Y     S  E  V  E  N  \0
b =   -1  0  0  0  0  0  0  0  0  1  2  3  4  5
```

Following the table *b*, we know that if we find a mismatch at $j = 11$, then we need to rewing j to $b[11] = 3$ to continue matching.

The text index i, on the other hand, will stay the same, and go forward by 1 if $b[j] = -1$.

KMP Algorithm – PseudoCode

```
char T[MAX_N], P[MAX_N];    int b[MAX_N], n, m;

void kmpPreprocess() {                         // Create the Back Array
  int i = 0, j = -1; b[0] = -1;
  while (i < m) {
     while (j >= 0 && P[i] != P[j]) j = b[j];
     i++; j++;
     b[i] = j; }}

void kmpSearch() {                             // Search the substring
  int i = 0, j = 0;
  while (i < n) {
     while (j >= 0 && T[i] != P[j]) j = b[j];
     i++; j++;
     if (j == m) {
        printf("P is found at index %d in T\n", i - j);
        j = b[j]; }}}
```

String Matching with the Z-Algorithm

Another alogirthm that performs string matcfhing in linear time is the **Z algorithm.**

The Z algorithm first makes a **Search String** $S = P +' S' + T$.
The Z algorithm next constructs a **Z array** of "prefix lengths".
For every index $i \in S$, $Z[i]$ is the size of the prefix of S that begins in i.

```
T = AASABAABAAT, P = AAB, S = P + '$' + T

... Build Z Array ...
S    = AAB$AASABAABAAT
Z[S]= X10021010310210
               ^
               String matched here. Z[i] = Len(P)
```

Z-Algorithm – Pseudocode

```
void Zarray(string S, int Z[]) {
    int n = S.length(); int L, R, k;
    L = R = 0;               // Prefix counters
    for (int i = 1; i < n; i++) {
        if (i > R) {         // Full search of prefix
            L = R = i;
            while (R < n && S[R] == S[R-L]) R++;
            Z[i] = R-L; R--;
        } else {             // Inside prefix candidate
            k = i-L;
            if (Z[k] < R-i+1) Z[i] = Z[k]; // no extension
            else {                         // prefix extension
                L = i;
                while (R < n && S[R] == S[R-L]) R++;
                Z[i] = R-L; R--;
}   }   }   }
```

Simulation: `https://personal.utdallas.edu/~besp/demo/John2010/z-algorithm.htm`

Z algorithm or KMP algorithm?

Should you use the Z algorithm or the KMP algorithm?

- Both algorithms have the same time complexity: $O(T + P)$
- Which algorithm is easier to understand?
  - KMP calculates a recursive suffix state machine for P;
  - Z-algorithm calculates a substring size array for T;

Part II: Strings and DP

String Algorithms with Dynamic Programming

Some string problems can be described as a **search problem.** In this section, we will introduce two string tasks that can be solved with DP algorithms:

- String Alignment/Edit Distance
- Longest Common Subsequence

It is interesting to note that substring matching is also a search problem, and that KMP / Z-algorithms can be seen as a kind of memoization.

String DP: String Alignment

The **String Alignment**[1] problem is defined as follows. Align two strings, A and B, with the maximum "alignment score":

- Character A[i] and B[i] match: do nothing, score +2
- Character A[i] and B[i] mismatch: replace A[i], score -1
- Insert a space in A[i]: score -1 (equals to delete B[i])
- Insert a space in B[i]: score -1 (equals to delete A[i])

```
   Original     non-optimal     optimal
A: ACAATCC    | A_CAATCC      | A_CAATCC
B: AGCATGC    | AGCATGC_      | AGCA_TGC
score:        | 2-22--2- = 4 | 2-22-2-2 = 7
```

[1] Also called Edit Distance or Levenhstein Distance, used by spellchecking algorithms!

String Alignment: Bottom Up DP

The **Complete Search** approach requires recursively testing each of the three options for each A[i] (Total cost: $O(3^n)$).

We can solve this in $O(n^2)$ using DP:

- $V(i,j)$: optimal score for prefix $A[1..i], B[1..j]$
- Start condition:
  - $V(0,0) = 0$ (Do nothing)
  - $V(i,0) = -1 \times i$, $V(0,j) = -1 \times j$ (delete A or B)
- Recurrence: $V(i,j) = \max(C_1, C_2, C_3)$, where
  - $C_1 = V(i-1, j-1) + \text{score}(A[i], B[j])$ score of match or mismatch;
  - $C_2 = V(i-1, j) + \text{score}(A[i], \_)$ delete $A[i]$;
  - $C_3 = V(i, j-1) + \text{score}(\_, B[j])$ delete $B[j]$;

String Alignment: Bottom Up DP

Simulation Matching AGCATGC and ACAATCC

- Recurrence: $V(i,j) = \max(C_1, C_2, C_3)$, where
  - $C_1 = V(i-1, j-1) + \text{score}(A[i], B[j])$ score of match or mismatch;
  - $C_2 = V(i-1, j) + \text{score}(A[i], \_)$ delete $A[i]$;
  - $C_3 = V(i, j-1) + \text{score}(\_, B[j])$ delete $B[j]$;

	_	A	G	C	A	T	G	C
_	0	-1	-2	-3	-4	-5	-6	-7
A	-1							
C	-2							
A	-3							
A	-4							
T	-5							
C	-6							
C	-7							

Problem 2: Longest Common Subsequence in Strings

Problem Definition

Given strings A and B, what is their longest common subsequence?

```
A   :   'ACAATCC'      - A_CAAT_CC
B   :   'AGCATGC'      - AGCA_TGC_
LCS:     AC AT C       - A_CA_T_C_ : ACATC
```

- We can solve LCS using a modification of String Aligment;
- Use String Alignment DP, with different scores:
  - Cost of Mismatch: $-\infty$
  - Cost of insert/deletion: 0
  - Cost of Matching: 1

Problem 3: Longest Palindrome

Problem Description

A **palindrome** is a string S where $S = \text{rev}(S)$. For example: MADAM.

Given a string T, what is the **longest palindrome** that you can create by deleting characters from T?

Examples:

- ADA**M** – ADA
- MADAM – MADAM
- NEVERODDOREVEN**ING** – NEVERODDOREVEN
- RACE**F1**CAR**FAST** – RACECAR

QUIZ: Can you solve with Full Search? String Alignment DP? Others?

Longest Palindrome

Problem Description

Given a string S of size up to $N = 1000$ characters, what is the longest palindrome that you can make by deleting characters from S?

DP Solution:

- State Table:
  - len(i,j) - The largest palindrome found between i and j
- Start Conditions:
  - If $l = r$ then $\text{len}(l, r) = 1$.
  - If $r = l + 1$ and $S[l] = S[r]$, $\text{len}(l, r) = 2$, else $\text{len}(l, r) = 1$.
- Transition:
  - If $S[l] = S[r]$, then $\text{len}(l, r) = 2 + \text{len}(l + 1, r - 1)$;
  - else $\text{len}(l, r) = \max(\text{len}(l + 1, r), \text{len}(l, r - 1))$

This DP has complexity $O(n^2)$

Longest Palindrome

Longest Palindrome DP: Diagonal Table Top Down

```
       len(l,r)                    len(l,r)         transition:
      final state               initial state     - If A[l] == A[r]: len(diag)+2
                                                  - If A[l] != A[r]: max(left,down)

  R A C E F 1 C A R           R A C E F 1 C A R

R 1 1 1 1 1 1 3 5 7         R 1 1
A   1 1 1 1 1 3 5 5         A   1 1
C     1 1 1 1 3 3 3         C     1 1
E       1 1 1 1 1 1         E       1 1
F         1 1 1 1 1         F         1 1
1           1 1 1 1         1           1 1
C             1 1 1         C             1 1
A               1 1         A               1 1
R                 1         R                 1
```

Part III: Suffix Tree, Array

Suffix Trie: Definition

Data structure used to find matching suffixes of multiple strings.

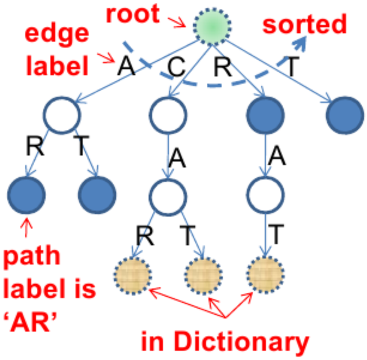
Suffix Trie for {'CAR','CAT','RAT'}

All Suffixes

1. CAR
2. AR
3. R
4. CAT
5. T
6. RAT
7. AT
8. T

Sorted, Unique Suffixes

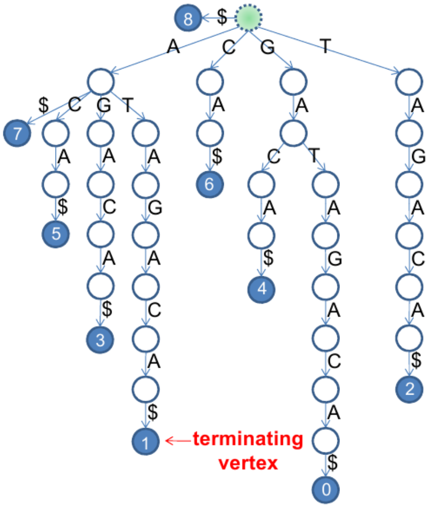
1. AR
2. AT
3. CAR
4. CAT
5. R
6. RAT
7. T (x2)



Suffix Trie: Counting the number of substrings of GATAGACA

Create all *n* suffixes:

i	suffix
0	GATAGACA$
1	ATAGACA$
2	TAGACA$
3	AGACA$
4	GACA$
5	ACA$
6	CA$
7	A$
8	$



Number of repeats of substring *m*:

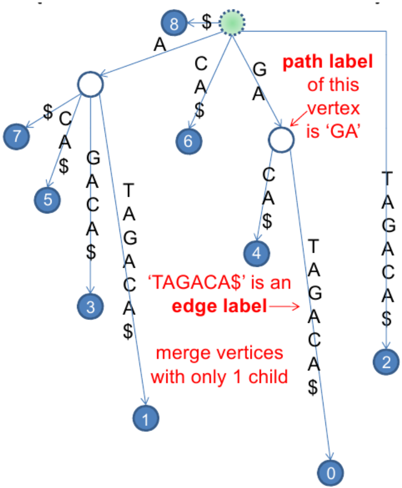
- 'A': 4 subtrees
- 'GA': 2 subtrees
- 'AA': 0 subtrees

Suffix Trie: Counting the number of substrings of GATACA

You can make the Suffix Tree better by merging the nodes that have a single child.

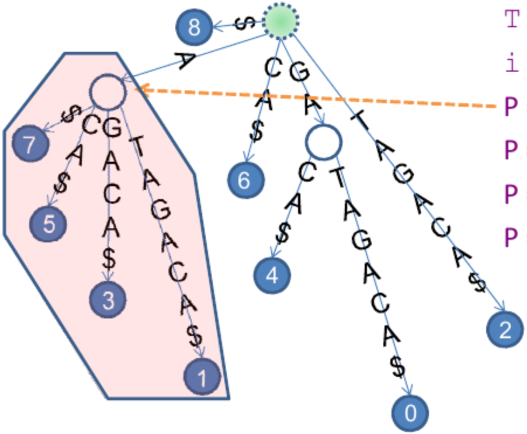
This data structure is useful for many algorithms.

i	suffix
0	GATAGACA$
1	ATAGACA$
2	TAGACA$
3	AGACA$
4	GACA$
5	ACA$
6	CA$
7	A$
8	$



Uses of a Suffix Tree 1: String Matching

Assuming that we have the Suffix Tree already built, we can find all occurrences of substring m in T in time $O(m + \text{occ})$, where occ is the number of occurrences.



T = 'GATAGACA$'

i = '012345678'

P = 'A' → Occurrences: 7, 5, 3, 1

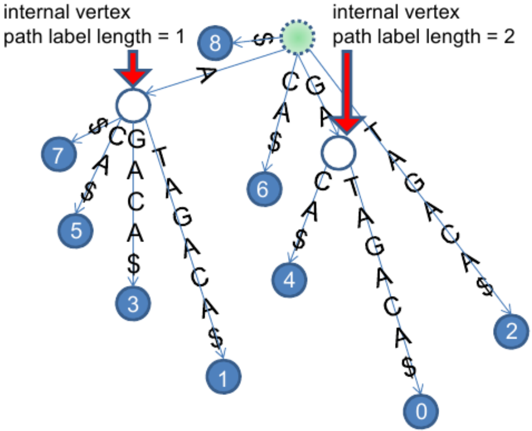
P = 'GA' → Occurrences: 4, 0

P = 'T' → Occurrences: 2

P = 'Z' → Not Found

Uses of a Suffix Tree 2: Longest Repeated Substring

- The LRS is the longest substring with number of occurrences > 2;
- The LRS is the deepest **internal node** in the tree;

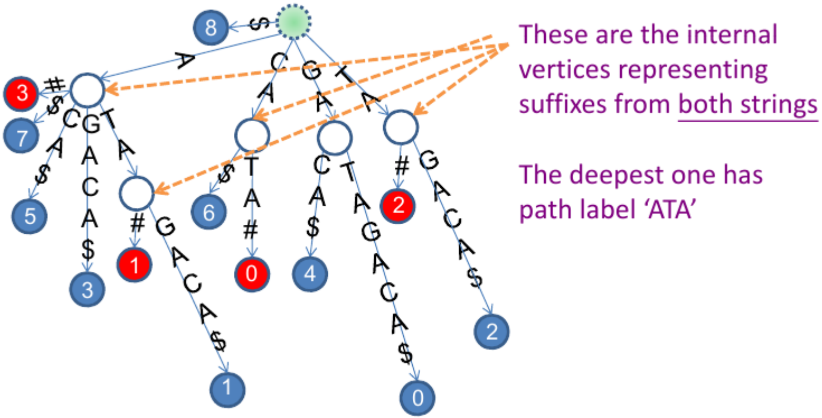


e.g. T = 'GATAGACA$'
The longest repeated substring is 'GA' with path label length = 2

The other repeated substring is 'A', but its path label length = 1

Uses of a Suffix Tree 3: Longest Common Substring

- Make a Suffix Tree of M and N combined, with a different ending character to each.
- The LCS is the deepest **internal node** that includes both ending characters.



Suffix Array

- The algorithms in previous slides are very efficient...
  ... if you have the suffix tree
- The suffix tree can be built in $O(n)$...
  ... but implementation is rather complex;
- In this course, we will see the Suffix Array;
- The Suffix Array is built in $O(n \log n)$...
  ... but the implementation is very simple!

I encourage you to study the implementation of the suffix tree by yourself!

Suffix Array Implementation Idea

- To make a Suffix array, make an array of all possible suffixes of T, and sort it;
- The order of the suffix array is the visit in preorder of the suffix tree;
- We can adapt all algorithms accordingly;

i	suffix
0	GATAGACA$
1	ATAGACA$
2	TAGACA$
3	AGACA$
4	GACA$
5	ACA$
6	CA$
7	A$
8	$

Sort $\rightarrow$

i	SA[i]	suffix
0	8	$
1	7	A$
2	5	ACA$
3	3	AGACA$
4	1	ATAGACA$
5	6	CA$
6	4	GACA$
7	0	GATAGACA$
8	2	TAGACA$

Suffix Array: Slow Implementation

Simple Implementation

```
#include <algorithm>
#include <cstdio>
#include <cstring>
using namespace std;
char T[MAX_N]; int SA[MAX_N],i,n;

bool cmp(int a, int b) { return strcmp(T+a, T+b) < 0; }
// O(n)

int main() {
  n = (int) strlen (gets(T));
  for (int i = 0; i < n; i++) SA[i] = i;
  sort (SA, SA+n, cmp); // O(n^2 log n) }
```

This implementation is too slow for strings bigger than 1000 characters.

Suffix Array: Better Implementation (1)

O(n log n) implementation using "ranking pairs/radix sort"

```
char T[MAX_N]; int n; int c[MAX_N];
int RA[MAX_N], tempRA[MAX_N], SA[MAX_N], tempSA[MAX_N];

void countingSort(int k) {
  int i, sum, maxi = max(300,n);                          //255 ASCII chars or n
  memset(c, 0, sizeof(c));
  for (i = 0; i < n; i++) c[i+k<n? RA[i+k] : 0]++
  for (i = sum = 0; i < maxi; i++)
    { int t = c[i]; c[i] = sum; sum += t; }               //frequency
  for (i = 0; i < n; i++)
    tempSA[c[SA[i]+k < n ? RA[SA[i]+k] : 0]++] = SA[i];
  for (i = 0; i < n; i++)                                 // update suffix array
    SA[i] = tempSA[i];
}

// ... continues next slide
```

Suffix Array: Better Implementation (2)

O(n log n) implementation using "ranking pairs/radix sort"

```
// ... continued from last slide
void constructSuffixArray() {
  int i, k, r;
  for (i = 0; i < n; i++) { RA[i] = T[i]; SA[i] = i;}
  for (k = 1; k < n; k <<=1) {
    countingSort(k); countingSort(0); tempRA[SA[0]] = r = 0;
    for (i = 1; i < n; i++)
      tempRA[SA[i]] =
          (RA[SA[i]] == RA[SA[i-1]] && RA[SA[i]+k] == RA[SA[i-1]+k]) ?
           r : ++r;
    for (i = 0; i < n; i++)
      RA[i] = tempRA[i];
    if (RA[SA[n-1]] == n-1) break;
  }
}
```

Using Suffix Array for String Matching:

- Do binary search two times: One to find the lower bound, one to find the upper bound;

Finding lower bound

i	SA[i]	Suffix
0	8	$
1	7	A$
2	5	ACA$
3	3	AGACA$
4	1	ATAGACA$
5	6	CA$
6	4	GACA$
7	0	GATAGACA$
8	2	TAGACA$

Finding upper bound

i	SA[i]	Suffix
0	8	$
1	7	A$
2	5	ACA$
3	3	AGACA$
4	1	ATAGACA$
5	6	CA$
6	4	GACA$
7	0	GATAGACA$
8	2	TAGACA$

Using Suffix Array for Longest Repeated Substring

Find the pair of indexes i and $i + i$ with longest common prefix.

i	SA[i]	LCP[i]	Suffix
0	8	0	$
1	7	0	A$
2	5	1	ACA$
3	3	1	AGACA$
4	1	1	ATAGACA$
5	6	0	CA$
6	4	0	GACA$
7	**0**	**2**	**GATAGACA$**
8	2	0	TAGACA$

Using Suffix Array for Longest Common Substring

Append strings *M* and *N* with different endings, and find LCS

i	SA[i]	LCP[i]	Owner	Suffix
0	13	0	2	#
1	8	0	1	$CATA#
2	12	0	2	A#
3	7	1	1	A$CATA#
4	5	1	1	ACA$CATA#
5	3	1	1	AGACA$CATA#
6	10	1	2	ATA#
7	**1**	**3**	**1**	**ATAGACA$CATA#**
8	6	0	1	CA$CATA#
9	9	2	2	CATA#
10	4	0	1	GACA$CATA#
11	0	2	1	GATAGACA$CATA#
12	11	0	2	TA#
13	2	2	1	TAGACA$CATA#

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