GB20602 - Programming Challenges

Week 9 - Computational Geometry

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Class Outline

- Functions for Points and Lines
- Functions for Triangles and Circles
- Functions for Polygons

Part I: Points and Lines

What are Computational Geometry Problems?

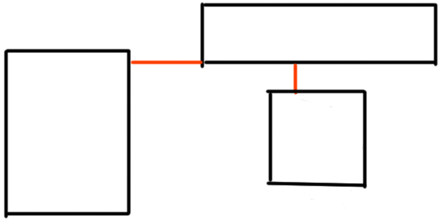
Computational Geometry problems involve the calculation of structures such as:

- Points;
- Lines;
- Circles;
- Triangles and Squares;
- General Poligons;

They can be very fun, but they are also a but troublesome to program.
Make sure to try the practice problems!

Example Geometry Problem

Input: N rectangles, described as $\{x, y, w, h\}$.
Output: Shortest length of axis-aligned lines to connect all rectangles;



How to solve?

- Calculate the axis aligned distance between every pair of rectangles.
  - Break the rectangle in 4 points;
  - Calculate distance between point and segment;
- Transform data into a graph;
- Use Minimum Spanning Tree to calculate Distance;

Be careful of special cases

Geometry problems usually have special cases. It is important to test your algorithm on paper before programming!

- Multiple points in the same position;
- Collinear points (three points in the same line);
- Vertical lines (bad tangent value, division by 0);
- Parallel Lines (bad intersection value);
- Intersection at end of a segment;
- etc;

Avoiding Precision Errors

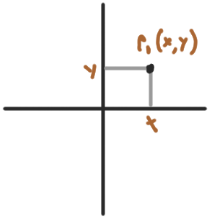
Geometry problems frequently require working with floating point numbers. Follow these hints to avoid precision errors:

- If possible, work with integers.
  - Try to do the floating point operations last, if possible.
- Use the Long Double data type;
- Use the following code for equality comparison:

```
if (float_1 == float_2)                 // Wrong!
if (fabs(float_1 - float_2) < EPS)      // Correct!
                                        // EPS = 10^-9
```

- Reduce the number of floating point operations.
  - Example: Instead of a/b/c, use a/(b*c);

Data Structure for Point




```
struct point_i {
  int x, y;  // integer coordinates
  point_i() { x = y = 0; }
  point_i(int _x, int _y) : x(_x), y(_y) {}
};

struct point {
  double x, y; // double coordinates
  point() { x = y = 0.0;}
  point(double _x, double _y) : x(_x), y(_y) {}
};
```

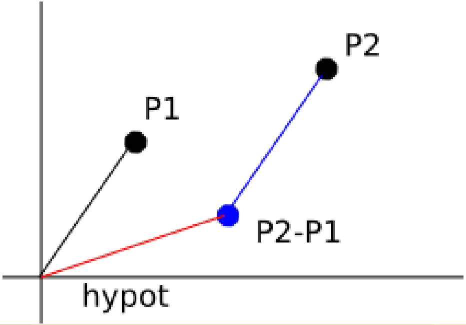
Overloading Point Operators

Allows sorting points with "sort()"

```
struct point { double x, y;
   point() { x = y = 0.0;
   point(double _x, double _y) : x(_x), y(_y) {}

   bool operator < (point other) const {           // Overloading "<"
      if (fabs(x - other.x) > EPS)
         return x < other.x;
      return y < other.y; }
   bool operator == (point other) const {          // Overloading "=="
      return (fabs(x - other.x) < EPS &&
              (fabs(y - other.y) < EPS)); }
   }

point a = point(1,2); point b = point(3,4);

if (!(a == b)) printf("Different\n");
```

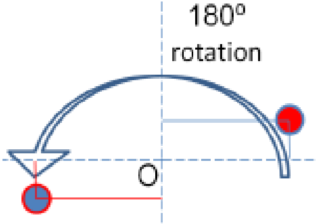
Point Distance

```
// Euclidean Distance: "normal" distance
#define hypot(dx,dy) sqrt(dx*dx + dy*dy)
double dist(point p1, point p2) { return hypot(p1.x - p2.x, p1.y - p2.y); }

// Taxicab Distance / Manhattan Distance : Distance on a grid
double taxicab(point p1, point p2) { return fabs(p1.x-p2.x) + fabs(p1.y-p2.y); }
```



Point Rotation

```
#define PI          3.14159265358979323846  // Pi constant
double PI = 2 * acos(0.0)                   // Better Pi
#define DEG_to_RAD(X) (X*PI)/180.0          // Don't confuse DEG and RAD

// suppose angle t is in degrees (0--360)
point rotate(point p, double t) {
   double rad = DEG_to_RAD(t);
   return point(p.x * cos(rad) - p.y * sin(rad),
                p.x * sin(rad) + p.y * cos(rad));}
```



$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix}$$

Data Structure for a Line

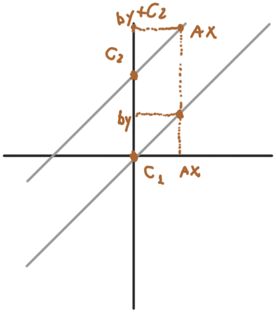
- $ax + by + c = 0$ (a,b,c) – useful for most cases
- $y = mx + c$ (m,c) – useful for angle manipulation
- x_0, y_0, x_1, y_1 (p_1, p_2) – harder to use, but common input

How to convert from two points to a line

```
struct line { double a,b,c; };

void pointsToLine(point p1, point p2, line &l) {
   if (fabs(p1.x - p2.x) < EPS {
      l.a = 1.0; l.b = 0.0; l.c = -p1.x; }
   else {
      l.a = -(double) (p1.y - p2.y)/(p1.x - p2.x);
      l.b = 1.0; l.c = -(double) (l.a*p1.x) - p1.y;}
}
```

Line Equality

- We define a line as $a, b, c \rightarrow (ax + by = c)$
- Two lines are parallel if their coefficients (a, b) are the same;
- Two lines are identical if all coefficients (a, b, c) are the same;




```
bool areParallel(line l1, line l2) {
   return (fabs(l1.a-l2.a) < EPS) && (fabs(l1.b-l2.b) < EPS); }

bool areSame(line l1, line l2) {
   return areParallel(l1,l2) && (fabs(l1.c - l2.c) < EPS); }
```

Line Intersection

The intersection point x_I, y_I can be found by solving:

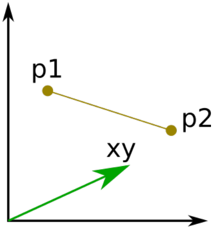
$$a_1 x_I + b_1 y_I + c_1 = 0$$
$$a_2 x_I + b_2 y_I + c_2 = 0$$

Remember that when we create a line i, we set $b_i = 0$ or $b_i = 1$

```
bool areIntersect(line l1, line l2, point &p) {
   if (areParallel(l1,l2)) return False;

   p.x = (l2.b * l1.c - l1.b * l2.c) /
         (l2.a * l1.b - l1.a * l2.b);

   // Test for vertical case:
   if (fabs(l1.b) > EPS)     p.y = -(l1.a * p.x + l1.c);
   else                      p.y = -(l2.a * p.x + l2.c);
   return True;
}
```

Segments and Vectors

- A **line segment** is a line with two endpoints (p_1, p_2);
- A **Vector** is a line segment with a direction;
  - Represented by its **direction** and **length**;
  - Direction: a point with distance 1 from $(0, 0)$
  - Length: an integer;
  - Used for movement, translation, speed, etc;




```
struct vec { double x, y;
    vec(double _x, double _y) : x(_x), y(_y) {} };

vec toVec(point a, point b) {
    return vec(b.x - a.x, b.y - a.y); }
vec scale(vec v, double s) {
    return vec(v.x * s, v.y * s); }
point translate(point p, vec v) {
    return point(p.x + v.x , p.y + v.y); }
```

Distance between point and line, point and segment

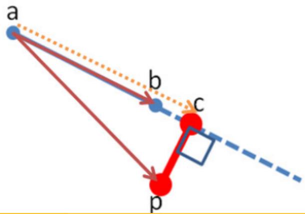
For a point p and a line l (represented by $\overline{ab}$), the distance between both is given by the segment pc, where c is the projection of p in l.

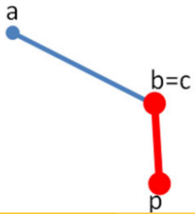
Calculating c:

- calculate scalar proj. u of $\vec{ap}$ in l.
- change magnitude of $\vec{ab}$ to u to obtain $\vec{ac}$.
- calculate distance between p and c.

Distance between p and $\overline{ab}$:

- calculate scalar proj. u of $\vec{ap}$ in l.
- if $u < 0$ or $u > |\vec{ab}|$, then the distance is $\min(d(a,p), d(b,p))$.
- else, calculate $\vec{ac}$ and calculate distance $d(p,c)$.





CODE: Distance between point and line

```
double dot(vec a, vec b) { return (a.x * b.x + a.y * b.y); } // dot product
double norm_sq(vec v) { return v.x * v.x + v.y * v.y; } // norm squared

// Given points a,b,p, calculate distance from p to line ab.
double distToLine(point p, point a, point b, point &c) {
  // point c: c = a + u * |ab|
  vec ap = toVec(a, p), ab = toVec(a, b);

  // dot product calculates size of ap in ab
  // norm square will calculate the scale to ab
  double u = dot(ap, ab) / norm_sq(ab);

  // translate a by u to find point c.
  c = translate(a, scale(ab, u));
  return dist(p, c);
}
```

CODE: Distance between point and segment

Same as before, but first we test if c is inside $\overline{ab}$.

```
double distToSegment(point p, point a, point b, point &c) {
  // next two lines is exact same as last slide
  vec ap = toVec(a, p), ab = toVec(a, b);
  double u = dot(ap, ab) / norm_sq(ab);

  // test if the magnitude $u$ is bigger or smaller than ab.
  if (u < 0.0) { c = point(a.x, a.y); // closer to a
                 return dist(p, a); }
  if (u > 1.0) { c = point(b.x, b.y); // closer to b
                 return dist(p, b); }

  // c is inside AB, same as last slide
  c = translate(a, scale(ab, u));
  return dist(p,c);
}
```

Part II: Triangle and Circle

Angle between segments

Given three points, a, b and o, we can calculate the angle between $\overline{oa}$ and $\overline{ob}$ using the dot product.

Given that: $oa \cdot ob = |oa| \times |ob| \times \cos(\theta)$, we have $\theta = \arccos(\frac{oa \cdot ob}{|oa| \times |ob|})$

```
#import <cmath>

// angle in radians (0..2*PI)
double angle(point a, point o, point b) {
  vec oa = toVector(o, a), ob = toVector(o, b);
  return acos(dot(oa, ob) / sqrt(norm_sq(oa) * norm_sq(ob)));
}
```

Left, Right and Collinear Points

Given a line defined by points p and q, we are interested in knowing if point r is on the left/right side of the line, or if the three ponts are collinear.

Let $\vec{pq}$ and $\vec{pr}$ be two vectors, the **cross product** $\vec{pq} \times \vec{pr}$ is a vector that is perpendicular to both vectors. The magnitude of the cross product is positive / zero / negative if $p \rightarrow q \rightarrow r$ is left turn / collinear / right turn.

```
double cross(vec a, vec b) { return a.x * b.y - a.y * b.x; }

bool ccw(point p, point q, point r) {
  return cross(toVec(p, q), toVec(p, r)) > 0; }

collinear(point p, point q, point r) {
  return fabs(cross(toVec(p, q), toVec(p, r))) < EPS;
```

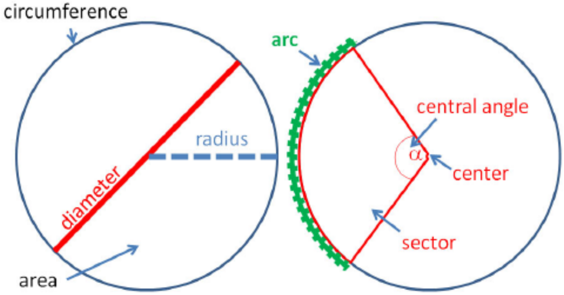
Circles

- A circle is stored as its center point c, and its radius r.
- The circle contains all points (x, y) where $(x-a)^2 + (y-b)^2 \leq r^2$
- No square root, so less chance of floating point errors.

Test if Point p is inside Circle – Integer Version

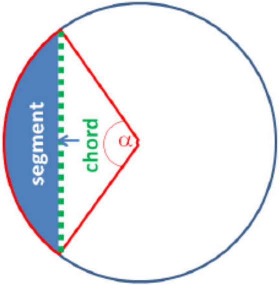
```
int insideCircle(point_i p, point_i c, int r) {
   int dx = p.x-c.x, dy = p.y-c.y;
   int Euc = dx*dx + dy*dy, rSq = r*r;
   return Euc < rSq ? 0 : Euc == rSq ? 1 : 2;
   // 0 - inside, 1 - border, 2- outside
}
```

Other circle properties



- **radius:** r, **diameter:** 2r, **circumference:** $2 \times \pi \times r$
- You can obtain π from the problem, or with $\pi = 2 \times \arccos(0)$
- Given a central angle α:
  - **Arc:** $r \times \alpha$ (if in rad) or $r \times \frac{\alpha}{360} \times 2\pi$ (if degrees)
  - **Sector:** $\frac{\alpha r^2}{2}$ (if in rad) or $2\pi r^2 \times \frac{\alpha}{360}$ (if degrees)

Other circle properties – chord



- **chord:** Line segment with two ends in the circle's border.
- If you know p_1, p_2 and c, you can find α from the "angle" function;
- If you know α and r, you can find the size of the chord by: $|p_1 p_2| = 2 \times r \times \sin(\alpha/2)$
  - **Quiz:** If you know a line and a circle, how do you find p_1 and p_2?
- If you know p_1, p_2, and r (but not α or c), you can find the center of the circle using the code in the next page;

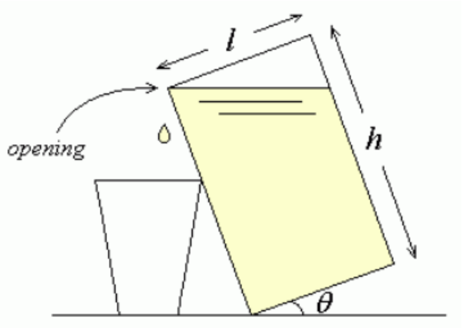
Circle Center from points and radius

You have two points p_1, p_2 that form a chord in a circle, and the radius of that circle. How do you find the center?

```
bool circle2PtsRad(point p1, point p2, double r, point &c) {
  double d2 = (p1.x - p2.x) * (p1.x - p2.x) +
              (p1.y - p2.y) * (p1.y - p2.y);
  double det = r * r / d2 - 0.25;

  if (det < 0.0) return false; // Can't make circle

  double h = sqrt(det);
  c.x = (p1.x + p2.x) * 0.5 + (p1.y - p2.y) * h;
  c.y = (p1.y + p2.y) * 0.5 + (p2.x - p1.x) * h;
  return true;
}
// to get the other center, reverse p1 and p2
```

Triangle Problem Example: Soya milk

- **Input**:
  The dimensions of a Milk box, and its inclination: l, w, h, θ
- **Output**:
  The amount of milk left in the box.



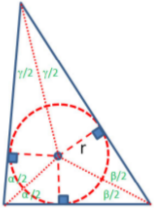
Triangle/Circle Problem Example: Bounding Box

- **Input**: Three points that are the vertices of a **regular polygon**, and number n of sides in the polygon;
- **Output**: Area of smallest **axis aligned** rectangle that bounds this polygon.

Triangle Basic Facts

- **Triangle Inequality**: If a, b, c are sides of a triangle, then $a + b > c; a + c > b; b + c > a$;
- **Perimeter, Semiperimeter**: $p = a + b + c$, $s = p/2$
- **Area**: $A = \frac{bh_b}{2}$
- **Area (Heron's formula)**: $A = \sqrt{s(s-a)(s-b)(s-c)}$
- **Triangulation**: Any 2D polygon can be decomposed into triangles;

Incircle of a Triangle



- An **Inscribed Circle (incircle)** is the largest circle that fits inside a triangle;
- The radius of the incircle is: $r = A/s$
- The center of the circle can be found by the intersection of two **angle bisectors**.

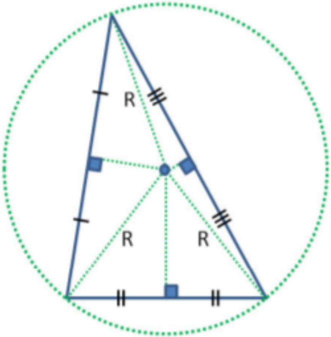
Radius of the Incircle: $r = \text{area}/s$

```
double rInCircle(double ab, double bc, double ca) {
  return area(ab, bc, ca) / (0.5 * perimeter(ab, bc, ca)); }

double rInCircle(point a, point b, point c) {
  return rInCircle(dist(a, b), dist(b, c), dist(c, a)); }
```

Finding the center of the Incircle of a triangle

```
int inCircle(point p1, point p2, point p3, point &ctr, double &r) {
  r = rInCircle(p1, p2, p3);
  if (fabs(r) < EPS) return 0;                // Test colinear points !!!
  line l1, l2;     // we calculate the intersect of two angle bisectors

  double ratio; point p;
  ratio = dist(p1, p2) / dist(p1, p3);
  p = translate(p2, scale(toVec(p2, p3), ratio / (1 + ratio)));
  pointsToLine(p1, p, l1);                    // bisector 1

  ratio = dist(p2, p1) / dist(p2, p3);
  p = translate(p1, scale(toVec(p1, p3), ratio / (1 + ratio)));
  pointsToLine(p2, p, l2);                    // bisector 2

  areIntersect(l1, l2, ctr);                  // find center (ctr)
  return 1;
}
```

Circumcircle of a Triangle



- The radius of the circumcircle in a triangle with sides a, b, c and area A is $R = \frac{abc}{4A}$;
- The radius R is also related to the **Law of Sines**:

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma} = 2R$$

- To find the center of the circumcircle:
  - Use a similar algorithm as the center of the incirle (last slide);
  - Instead of angle bisectors, use **perpendicular bisectors**;

Part III – Polygons and Convex Hull

Polygons – Definition and data structure

A polygon is a plane figure bounded by a finite sequence of line segments.

Polygon Representation

- In general, we store an array of points of the segments;
- We want to sort the points in CW or CCW order;
- Add the first point at the end of the array to avoid special cases;

```
// 6 points, entered in counter clockwise order;
vector<point> P;
P.push_back(point(1, 1)); // P0
P.push_back(point(3, 3)); // P1
P.push_back(point(9, 1)); // P2
P.push_back(point(12, 4)); // P3
P.push_back(point(9, 7)); // P4
P.push_back(point(1, 7)); // P5
P.push_back(P[0]); // important: loop back
```

Characteristics of a Polygon

Perimeter of a Poligon – add the distances of the segments

```
double perimeter(const vector<point> &P) {
  double result = 0.0;
  for (int i = 0; i < (int)P.size()-1; i++)
     // remember: P[0] = P[P.size()-1]
     result += dist(P[i], P[i+1]);
  return result; }
```

Area of a Poligon – half of the determinant of the XY matrix of segments

```
double area(const vector<point> &P) {
  double result = 0.0, x1, y1, x2, y2;
  for (int i = 0; i < (int)P.size()-1; i++) {
    x1 = P[i].x; x2 = P[i+1].x;
    y1 = P[i].y; y2 = P[i+1].y;
    result += (x1 * y2 - x2 * y1); }
  return fabs(result) / 2.0; }
```

Testing if a Polygon is Convex

- A convex polygon has no "holes";
- For any 2 points p_1, p_2 inside the polygon, segment is inside polygon too.

Easier Convex Testing: Every angle turns the same way

```
bool isConvex(const vector<point> &P) {
  // Returns true if every 3 neighb vertices turn the same way;
  int sz = (int)P.size();
  if (sz <= 3) return false; // Not a polygon

  bool isLeft = ccw(P[0], P[1], P[2]); // described earlier

  for (int i = 1; i < sz-1; i++)
    if (ccw(P[i],P[i+1],P[(i+2) == sz ? 1 : i+2]) != isLeft)
      return false;              // not same direction as isLeft.

  return true;
}
```

Polygon – Test point inside the polygon

We can use the same idea to test if a point is inside the polygon: The direction of the point in relation to every edge should be the same.

Winding Algorithm Code for point in polygon detection

```
bool inPolygon(point pt, const vector<point> &P) {
  if ((int)P.size() == 0) return false;
  double sum = 0;

  for (int i = 0; i < (int)P.size()-1; i++) {
    if (ccw(pt, P[i], P[i+1]))
      sum += angle(P[i], pt, P[i+1]);        //left turn/ccw
      else sum -= angle(P[i], pt, P[i+1]);   //right turn/cw
  }

  return fabs(fabs(sum) - 2*PI) < EPS;
}
```

QUIZ: What happens if the point is at an edge segment?

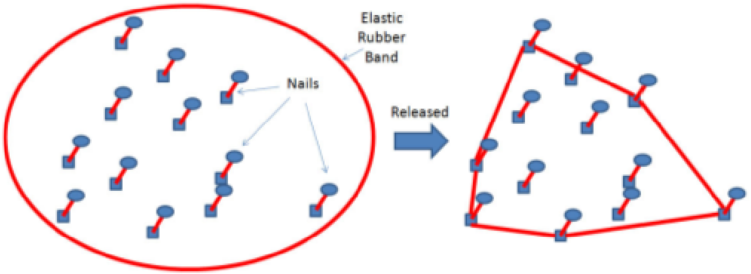
Polygon – Cutting

To cut *P* along a line *AB*, we separate the points in *P* to the left and right of the line.

```
point lineIntersectSeg(point p, point q, point A, point B) {
  double a = B.y-A.y; double b = A.x - B.x; double c = B.x*A.y - A.x*B.y;
  double u = fabs(a*p.x + b*p.y + c); double v = fabs(a*q.x + b*q.y + c);
  return point((p.x*v + q.x*u)/(u+v), (p.y*v + q.y*u)/(u+v)); }

vector<point> cutPolygon(point a, point b, const vector<point> &Q){
  vector<point> P;
  for (int i = 0; i < (int)Q.size(); i++) {
    double left1 = cross(toVec(a, b), toVec(a, Q[i])), left2 = 0;
    if (i != (int)Q.size()-1)
      left2 = cross(toVec(a, b), toVec(a, Q[i+1]));
    if (left1 > -EPS)
      P.push_back(Q[i]);                        //Q[i] is on the left of ab
    if (left1*left2 < -EPS)                     //edge (Q[i], Q[i+1]) crosses line ab
      P.push_back(lineIntersectSeg(Q[i], Q[i+1], a, b)); }
  if (!P.empty() && !(P.back() == P.front()))
    P.push_back(P.front());                     // make P's first point = P's last point
  return P; }
```

Polygon – Convex Hull

- A common problem: Given a set of points S, what is the **smallest convex polygon** that includes all points in S?
- One way to find the Convex Hull: for each point $p \in S$, determine if the point is at the edge of the polygon (in the hull) or inside the polygon (not in the hull).
- We will introduce the $O(n \log n)$ algorithm "Graham's Scan"



Polygon – Graham's Scan

Helper Functions – sort two points based on their angle against the X axis

```
point pivot(0, 0);

bool angleCmp(point a, point b) {
  // special case: if collinear, choose closet to pivot;
  if (collinear(pivot, a, b)) // special case
    return dist(pivot, a) < dist(pivot, b);

  // calculate angle against the X axis:
  double d1x = a.x - pivot.x, d1y = a.y - pivot.y;
  double d2x = b.x - pivot.x, d2y = b.y - pivot.y;

  return (atan2(d1y, d1x) - atan2(d2y, d2x)) < 0;
}
```

Polygon – Graham's Scan

Convex Hull – Initializing the algorithm

```
vector<point> CH(vector<point> P) {
  int i, j, n = (int)P.size();
  // Special Case: Polygon with 3 points
  if (n <= 3) {
    if (!(P[0]==P[n-1])) P.push_back(P[0]);
    return P; }

  // Find Initial Point: Low Y then Right X
  int P0 = 0;
  for (i = 1; i < n; i++)
    if (P[i].y < P[P0].y ||
        (P[i].y == P[P0].y && P[i].x > P[P0].x))
      P0 = i;
  point temp = P[0]; P[0] = P[P0]; P[P0] = temp;
```

Polygon – Graham's Scan

Convex Hull – More initialization

```
// second, sort points by angle with pivot P0
pivot = P[0];
sort(++P.begin(), P.end(), angleCmp);

// S holds the Convex Hull
// We initialize it with first three points
vector<point> S;
S.push_back(P[n-1]);
S.push_back(P[0]);
S.push_back(P[1]);

// We start on the third point
i = 2;
```

Polygon – Graham's Scan

Convex Hull – Main Loop

Now that we selected a pivot and sorted the points, we test every three points (following the sort) if they are in the convex hull.

```
  while (i < n) {
    j = (int) S.size() - 1;
    // If the next point is left of CH, keep it.
    // Else, pop the last CH point and try again.

    if (ccw(S[j-1], S[j], P[i]))
      S.push_back(P[i++]);
    else
      S.pop_back();
    }
  return S;
}                  // End Graham's Scan CH
```

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