GB20602 - Programming Challenges

Week 10 - Final Problem Remix

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Lesson Outline

The final week is a **Remix** week:

- The problems revisit topics from week 1-9;
- Composite problems (2+ topics together);

Topics in this material:

- Choosing the right DP table;
- DP for Travelling Salesman Problem (TSP);
- Composite Problems (BSTA + something else);

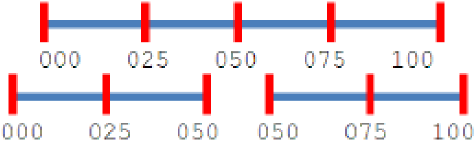
Part I – More DP

Choosing the right DP table size – "Cutting Sticks"

Problem Description

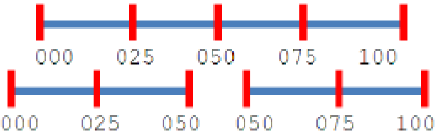
- In a stick of length l $(1 \leq l \leq 1000)$
- Make N cuts at positions cuts $= \{c_1, c_2, \ldots, c_N\}$ $(1 \leq N \leq 50)$
- The cost of a cut is the size of the sub-stick that you cut.
- What order of cuts minimize the cost?

Example: $l = 100, N = 3,$ cuts $= \{25, 50, 75\}$



- Sequence 1: 25, 50, 75. Cost: 100 + 75 + 50 = 225
- Sequence 2: 50, 25, 75. Cost: 100 + 50 + 50 = 200

Cutting Sticks – Thinking about the Problem



- Sequence 1: 25, 50, 75. Cost: 100 + 75 + 50 = 225
- Sequence 2: 50, 25, 75. Cost: 100 + 50 + 50 = 200

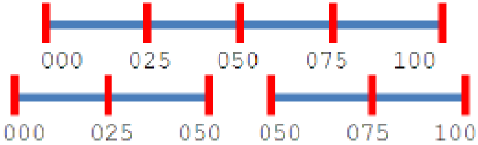
Part 1 – consider full search

- What is the algorithm for a full search?
- What is the complexity of this algorithm? And the maximum time?

Part 2 – Consider DP

- This problems smells of **DP**. (Find Maximum, several choices)
- Think about the states (DP table) and how to change between them (transition)

utting Sticks – Top Down DP (Recursive Function)



- Sequence 1: 25, 50, 75. Cost: 100 + 75 + 50 = 225
- Sequence 2: 50, 25, 75. Cost: 100 + 50 + 50 = 200

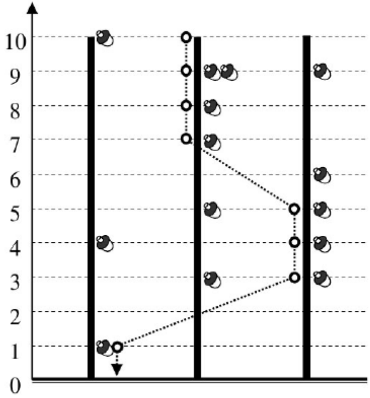
Recurrence

Let's think of a Top-down DP based on a recursive function:

- $A = \{0, c_1, c_2, \ldots c_N, N+2\}$ is the set of all cutting points, plus the start and end point.
- $\text{cost}(a_i, a_j) = \text{dist}(a_i, a_j) + \min_{i \leq k \leq j}(\text{cost}(a_i, a_k) + \text{cost}(a_k, a_j))$
- $\text{cost}(a_i, a_i) = 0$

This requires at most a (N, N) DP table for memoization, and O(N) for each iteration.

DP Problem 2 – Acorn

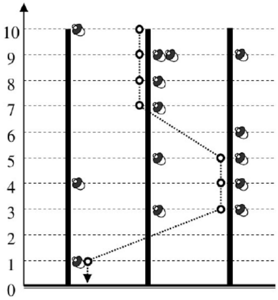


- Begin at the top of a tree, and get the maximum number of acorns.
- You can go down **1 height** on the tree.
- OR **change tree** for the cost of **f height** (In this figure, $f = 2$)

- Number of trees: $1 \leq T \leq 2000$
- Height of trees: $1 \leq H \leq 2000$
- Length of fall : $1 \leq f \leq 500$

- First, it is worth to think about the full search size;
- But this problem smells of DP – can you think of a **transition** and a **state table**?

ACORN – Simple Recurrence



Simple Recurrence

- acorn[t_i][h] – number of acorns in tree t_i at height h
- $\text{cost}(t_i, 0) = \text{acorn}[t_i][0]$
- $\text{cost}(t_i, j) = \text{acorn}[t_i][j] + \max_{k \neq t_i}(\text{cost}(t_i, j-1), \text{cost}(t_k, j-f))$
  (Don't forget to check $j - f < 0$)
- Final cost: $\max_{1 \leq i \leq T}(\text{cost}[t_i][H])$

QUIZ: What is the problem with this recurrence?

ACORN – Finding a Better DP table

The DP table of last slide is A[H][T], with size $2000 * 2000 = 4M$. Each function call is $O(H * T * T)$, so total complexity is $4M * 2000 = 8B$

- Cost of changing tree is constant for any two trees.
- It is not necessary to keep all trees, only the best.

Better Recurrence – $O(H * T)$

We use the table dp[H] which contains the best solution at height H.

- $\text{dp}[0] = \max_{1 \leq j \leq T} \text{acorn}[j][0]$
- $\text{acorn}[j][i] += \max(\text{acorn}[j][i-1], \max[i-f])$
- $\text{dp}[i] = \max_{1 \leq j \leq T}(\text{acorn}[j][i])$

Part II – DP for Travelling Salesman Problems

TSP Problem – Blackbeard the Pirate

Blackbeard has to collect all treasures (up to 10) in the island. He cannot cross water or trees, and he must stay 1 square away from natives.

Black beard speed is 1 square / second. How long does it take to get all treasure and return to the ship?

```
10 10
~~~~~~~~~~           ~ -- Water, can't cross
~~!!!###~~           # -- Trees, can't cross
~##...###~           ! -- Treasure, get these!
~#....*##~           . -- Just sand
~#!..**~~~           * -- Natives, don't get close here.
~~....~~~~           @ -- Landing point, start and return here.
~~~....~~~
~~..~..@~~           The solution for this case is: 32
~#!.~~~~~~           QUIZ: How to solve this problem?
~~~~~~~~~~
0 0
```

Blackbeard the Pirate – Problem Idea

One way to solve this problem is to break it into two parts:

1. Create a treasure path graph from the input map
2. Small number of treasures: Find TSP of treasure path

```
10 10
~~~~~~~~~~        ~ -- Water, can't cross
~~!!!###~~        # -- Trees, can't cross
~##...###~        ! -- Treasure, get these!
~#....*##~        . -- Just sand
~#!..**~~~        @ -- Landing point, return here.
~~....~~~~
~~~....~~~
~~..~..@~~
~#!.~~~~~~
~~~~~~~~~~
0 0
```

Blackbeard – Extracting the graph

```
10 10
~~~~~~~~~~   ##########   # -- Obstacle (waters and trees)
~~!!!###~~   ##345#####   X -- Obstacles (natives, just for clarity)
~##...###~   ###..X####   . -- Path
~#....*##~   ##..XXX###   0-9 -- Nodes
~#!..**~~~   ##2.XXX###
~~....~~~~   ##..XX####
~~~....~~~   ###....###
~~..~..@~~   ##..#..0##
~#!.~~~~~~   ##1.######
~~~~~~~~~~   ##########
0 0
```

- We can simply the graph into obstacles, paths and goals
- We are only interested in the treasures and goals, so how to find the pairwise distance between treasures?
- Answer:
- The result is a small graph with **at most** 11 vertices.

Blackbeard – Extracting the graph

```
10 10
~~~~~~~~~~   ##########   # -- Obstacle (waters and trees)
~~!!!###~~   ##345#####   X -- Obstacles (natives, just for clarity)
~##...###~   ###..X####   . -- Path
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~~....~~~~   ##..XX####
~~~....~~~   ###....###
~~..~..@~~   ##..#..0##
~#!.~~~~~~   ##1.######
~~~~~~~~~~   ##########
0 0
```
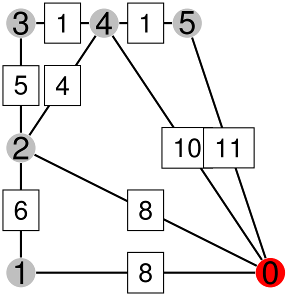

- We can simply the graph into obstacles, paths and goals
- We are only interested in the treasures and goals, so how to find the pairwise distance between treasures?
- Answer: BFS from each treasure/start point
- The result is a small graph with **at most** 11 vertices.

Blackbeard – Extracting the graph

```
##########
##345#####
###..X####
##..XXX###
##2.XXX###
##..XX####
###....###
##..#..0##
##1.######
##########
```

```
BFS from each vertex
-------------------->
Not all paths shown
```



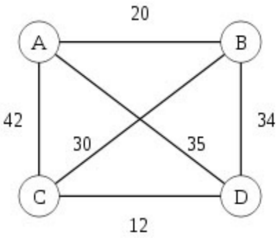
How do we find the minimal cycle starting in **S**, passing by all vertices?

The Traveling Salesman Problem (TSP)

Problem Definition

You have n cities, and their distances. Calculate the cost of the tour that starts and ends at a city s, passing through all other cities.

Exactly what we need! The path for all treasure!



	A	B	C	D
A	0	20	42	35
B	20	0	30	34
C	42	30	0	12
D	35	34	12	0

In the graph above, we have $n = 4$ cities and the minimal tour is A-B-C-D-A, with cost $20 + 30 + 12 + 35 = 97$.

QUIZ: What is the cost of solving TSP with complete search?

Characteristics of TSP

- A complete search for TSP costs $O(n! * n)$ – Search each city permutation.
- TSP is a **NP-hard** problem. This means that there is no known polinomial algorithm to solve it.
- However! For small values of n, there are some hacks to make the solution faster.

DP approach to TSP

The complete search for the TSP contains many repeated subsolutions:

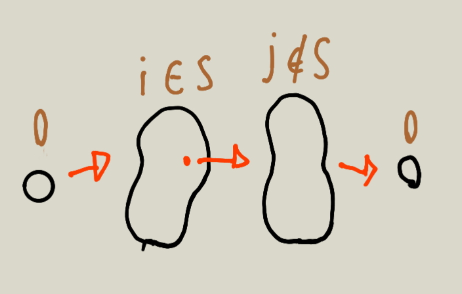
- S–A–B–C–. . .–S
- S–B–A–C–. . .–S

The minimum cost for C–. . .–S is the same. Can we use *memoization* to remember this cost?

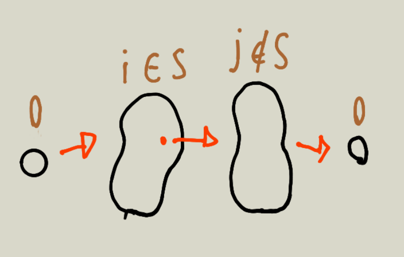
DP approach to TSP (1) – Idea

- We have already visited the cities $S = \{s_1, s_2, \ldots, s_n\}, s_i \neq 0$
- We are **now** in city $s_k \in S$
- What is the shortest path from s_k to 0, that passes in all cities $s_j \notin S$?

DP induction: shortest_path(S,s_k)



DP approach to TSP (2) – DP Recurrence



- We have visited all cities, and must return to the origin:
  shortest_path(S_{all}, s_k) = $D(s_k, 0)$
- We have visited some cites (S), and must find the next one:
  shortest_path(S, s_k) = $\min_{s_i \notin S}(D(s_k, s_i) +$ shortest_path($S \cup s_i, s_i$))
- Initial call:
  shortest_path($S = \emptyset$,0)

DP approach to TSP (3) – Implementation

- Our DP table is (*all sets*,*all cities*) – $2^n * n$
- We can represent a set of cities using a bitmask
- At each call, we loop through all cities, so the complexity is ($O(2^n * n^2)$)

- TSP using full search: $O(n! * n)$
- TSP using DP: $O(2^n * n^2)$ – Still low, but much better!

DP approach to TSP (4) – Sample Code

```
int dp[n][1<<n] = -1
start = 0

visit(p,v):
   if (v == (1<<n) - 1):
      return cost[p][start]
   if dp[p][v] != -1
      return dp[p][v]

   tmp = MAXINT
   for i in n:
       if not(v && (1 << i):
           tmp = min(tmp,
                     cost[p][i] + visit(i, v | (1<<i)))

   dp[p][v] = tmp
   return tmp
```

Part III – Composite Problems

Composite Problems

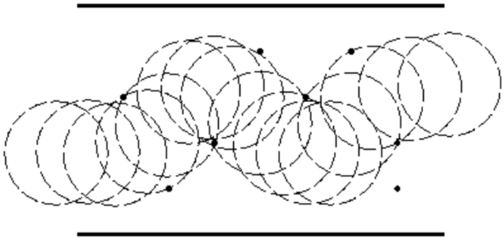
Until now, we could solve each problem with 1 algorithm.

But many interesting problems combine multiple algorithms!

A common combination is **Binary Search + Solve smaller problem**:

- Binary Search + Geometry Problem;
- Binary Search + Graph Search;
- Binary Search + DP;
- Binary Search + Greedy;
- etc...

UVA 295 – Fatman!

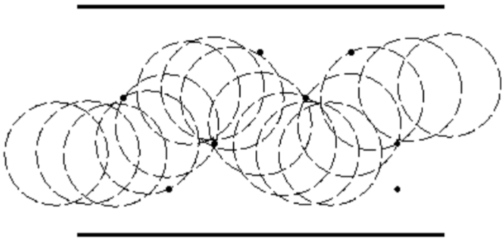


Problem Description

Find the **maximum diameter D of the circle** that can pass the corridor.

- The corridor has length L and width W;
- The corridor has $0 \leq N \leq 100$ obstacles, represented by (x_i, y_i);
- Obstacles are **points** with $0 \leq x_i \leq L, 0 \leq y_i \leq W$;

UVA 295 – Fatman – Breaking up the problem



Fatman Image from CPBook4 (Steven Halim)

One way to solve some problems is to break them down into smaller components.

1. Is it possible for a circle of radius $R, 0 \leq R \leq W$ to pass?
2. What is the maximum R that can pass?

QUIZ: Assume that (1) is "fast enough", how do we solve (2)?

UVA 295 – Fatman – Binary Search

- Is it possible for a circle of size $0 \leq R \leq W$ to pass?
- What is the maximum R that can pass?

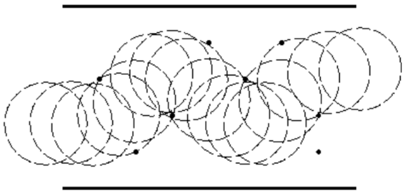
If we have a "fast" function $T(R)$ that tests if R can pass or not, we can use **Binary Search** to find the maximum R that pass:

1. Start with $R_l = 0, R_h = W$, Test $T(R_l + R_h/2)$;
2. If fails, $R_h = R_l + R_h/2$, else $R_l = (R_l + R_h)/2$; repeat $T(R_l + R_h/2)$.
3. Repeat until $R_h - R_l < 0.0001$.

This requires $\log_2(100 * 10000) = 20$ operations.

QUIZ: How can we test $T(R)$ “fast enough”?

UVA 295 – Fatman – Squeezing through



- R can pass between two objects *i* and *j* if $euclid(i,j) \geq R$
- R can pass between an object *i* and a wall if $y_i \geq R || y_i \leq W - R$

Algorithm for T(R)

- Create a Graph *G* where the obstacles and walls are vertices;
- If R can **not** pass between *i* and *j*, add an Edge E_{ij};
- If there is a **path** between both walls, R cannot pass;

UVA 295 – Fatman – Squeezing through

T(R) sample code – part 1, construct graph

```
def test(R):
  nb = []                       # list of neighbor list
  for i in range(len(N)+2): nb[i] = list()

  for i in range(len(N)):  # N is list (x,y) of obstacles
    if (N[i][1] < R): nb[0].append(i+1)
    if (W - N[i][1] < R): nb[len(N)+1].append(i+1)
    if (i+1) in nb[0] and (i+1) in nb[len(N)+1]: return 0  # quick check 1
  if not (len(nb[0]) and len(nb[len(N)+1])): return 1      # quick check 2

  for i in range(len(N)):
    for j in range(len(N)):
      if dist(N[i],N[j]) < R: nb[i+1].append(j+1)
  ... next we test the graph ...
```

UVA 295 – Fatman – Squeezing through

QUIZ: What is the total cost of this approach?

T(R) sample code – part 2, testing the graph

```
def test(R):
  nb = []                      # list of neighbor list
  for i in range(len(N)+2): nb[i] = list()
  for i in range(len(N)):    ... border test ...
  for i in range(len(N)):
    for j in range(len(N)): ... build graph ...

  curnode = 0; visited = list(); tovisit = list()
  while 1: # DFS
    if (curnode == len(N)+1) return 0    # reached wall
    visited.add(curnode)
    for i in nb[curnode]: tovisit.append(i)
    while(curnode in visited):
       if not (len(tovisit)): return 1  # not reached wall
       curnode = tovisit.pop()
```

UVA 714 – Copying books

Problem Description

- There are M books and K scribes ($1 \leq K \leq M \leq 500$).
- The each book has p_i pages ($1 \leq p_i \leq 1000000$)
- Assign books to each scribe, and **minimize** maximum job.
- Books must be assigned in blocks.

```
9 3
Input 1: 100 200 300 400 500 600 700 800 900
Output 1: 100 200 300 400 500 / 600 700 / 800 900 (max 1700)

5 4
Input 2: 100 100 100 100 100
Output 2: 100 / 100 / 100 / 100 100 (max 200)
```

- QUIZ: Describe the full search (and complexity)
- QUIZ: Describe a better algorithm?

UVA 714 – Copying books – Decomposition approach

```
9 3
Input 1: 100 200 300 400 500 600 700 800 900
Output 1: 100 200 300 400 500 / 600 700 / 800 900 (max 1700)

5 4
Input 2: 100 100 100 100 100
Output 2: 100 / 100 / 100 / 100 100 (max 200)
```

- Someone has probably suggested DP. It is certainly possible.
- We could also use “Binary Search + Test” from the last problem:
  - Binary search the maximum cost (100000*500 = 26 comparisons)
  - Test if the maximum cost is possible (T(max))
  - QUIZ: What is a “fast enough” algorithm for T(max)?

UVA 714 – Copying books – Testing a solution

```
9 3
Input 1: 100 200 300 400 500 600 700 800 900
Output 1: 100 200 300 400 500 / 600 700 / 800 900 (max 1700)
```

One possible Test: Greedy Algorithm to test Maximum M

```
def test(M):
  scribe = 0; book = 0;
  while scribe < K:
    sum = 0
    while sum + page[book] < M:
      sum += page[book]; book += 1
      if book == M: return 1   # assigned all books
    scribe ++
  return 0                     # did not assign all books
```

UVA 1079 – A careful Approach

Problem Description

- Choose the landing time t_i for $2 \leq N \leq 8$ planes;
- The minimum gap $|t_i - t_j|$ must be as large as possible;
- Each plane i has a maximum and minimum allowed landing time: $0 \leq \text{min}_i \leq t_i \leq \text{max}_i \leq 1440$

```
Input:                     Solution:
3 planes                   Maximum Minimum Gap: 7.5 minutes
1- 0 to 10                 P1 - Arrive at 0
2- 5 to 15                 P2 - Arrive at 7.5
3- 10 to 15                P3 - Arrive at 15
```

How do you solve it? (1- Binary search the GAP, 2- full search plane order, 3-greedy for landing time)

The End – Have a nice summer!

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